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Switching-Converter Dynamic Analysis with Fast Analytical Techniques

Overview and applications

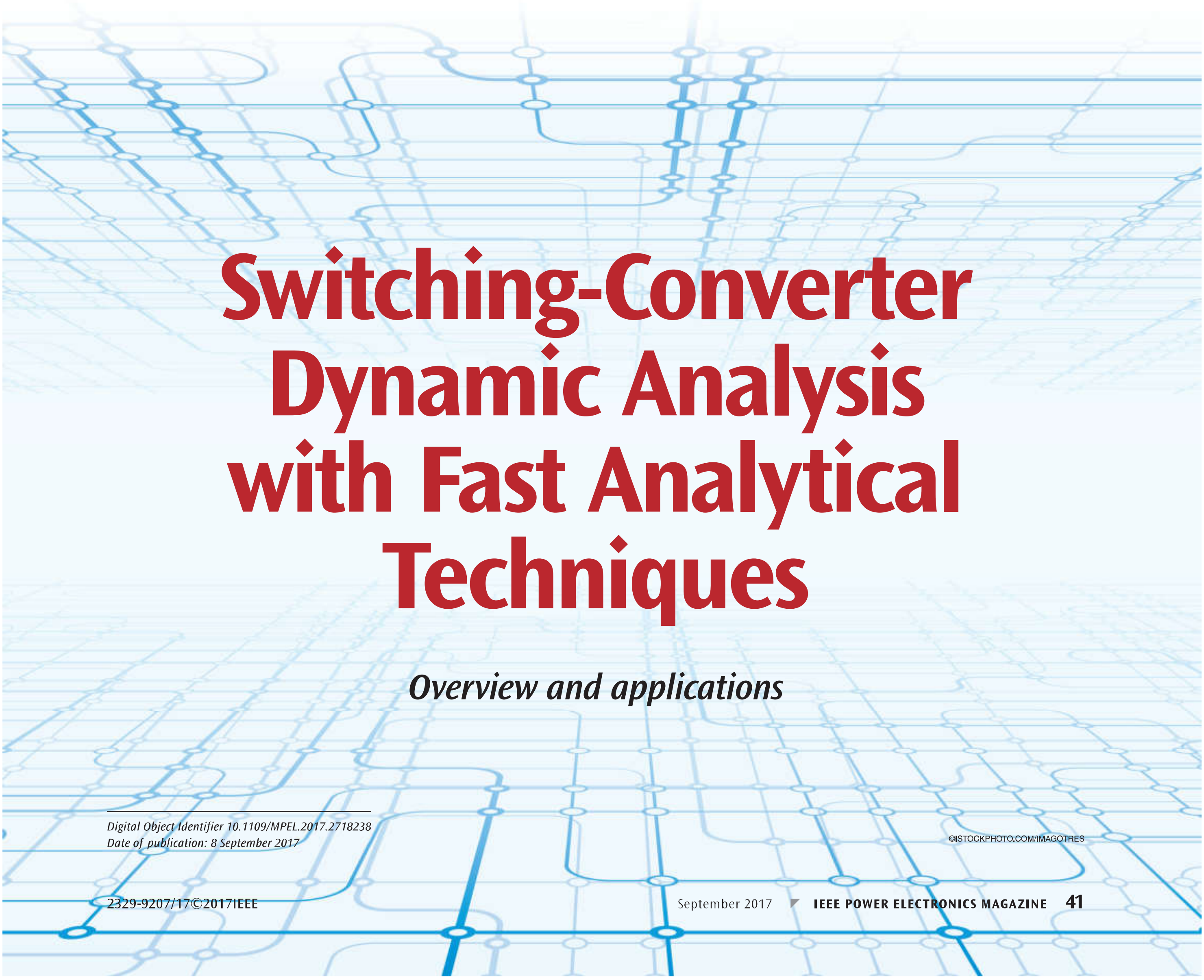
by Christophe Basso

If mesh-node analysis lends itself well to solving transfer functions of electrical circuits, obtaining a meaningful symbolic formula at once is often impossible and requires extra efforts to get it. Applying classical analysis techniques to obtain a so-called low-entropy expression—implying the factored form in which you distinguish gains, poles, and zeros—can often lead to algebraic paralysis, as R.D. Middlebrook commented in his foundational papers [1], [2]. This is where the fast analytic circuits techniques (FACTs) can help you build on what you learned during your years at university and extend the reach to drastically simplify analyses. By using FACTs, you not only gain in execution speed, but the final result appears in a well-ordered polynomial form, often without the need of further factoring efforts [3], [4].

This article begins with an introduction of FACTs, later applied to determine the control-to-output transfer function of a switching converter. The subject is vast, and we will only scratch the surface here; hopefully, encouraging you to dig further into the subject. We have selected the voltage-mode, coupled-inductors, single-ended primary inductor converter (SEPIC) operated in the discontinuous conduction mode (DCM). The pulsewidth modulation (PWM) switch [5] will be used to form the small-signal model.

A Quick Introduction to Fast Analytical Techniques

The basic principle behind FACTs lies in the determination of the circuit time constants, $\tau = RC$ or $\tau = L/R$, when the network under study is observed in two different conditions: 1) when the excitation signal is reduced to zero and 2) when the response is nulled. By using this technique, you will appreciate how quickly and intuitively one can determine a particular transfer function. The analysis techniques based on this method reach back several decades, as documented in [6] and [7].



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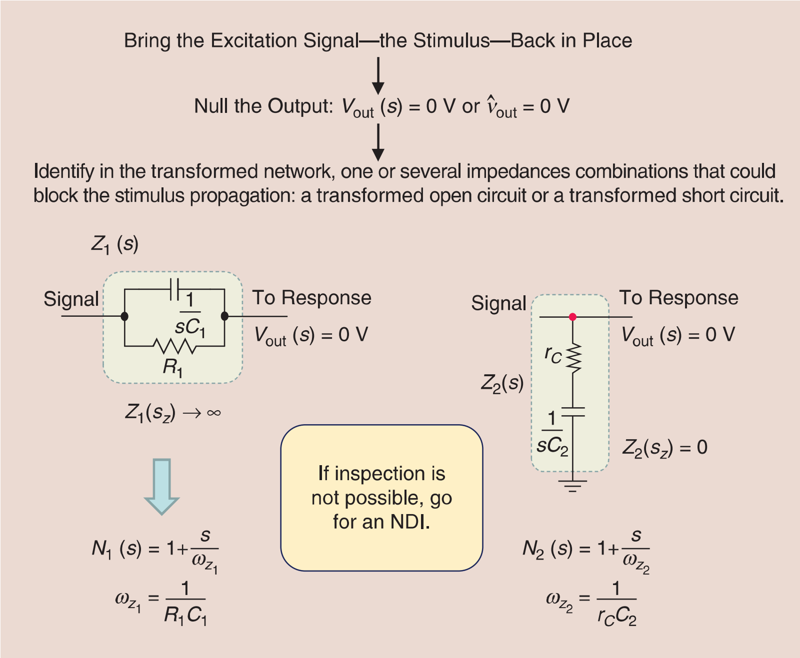

FIG 1 This simple flowchart will guide you in determining zeros in the simplest way. When inspection does not work, you will need to go for a null double injection (NDI).

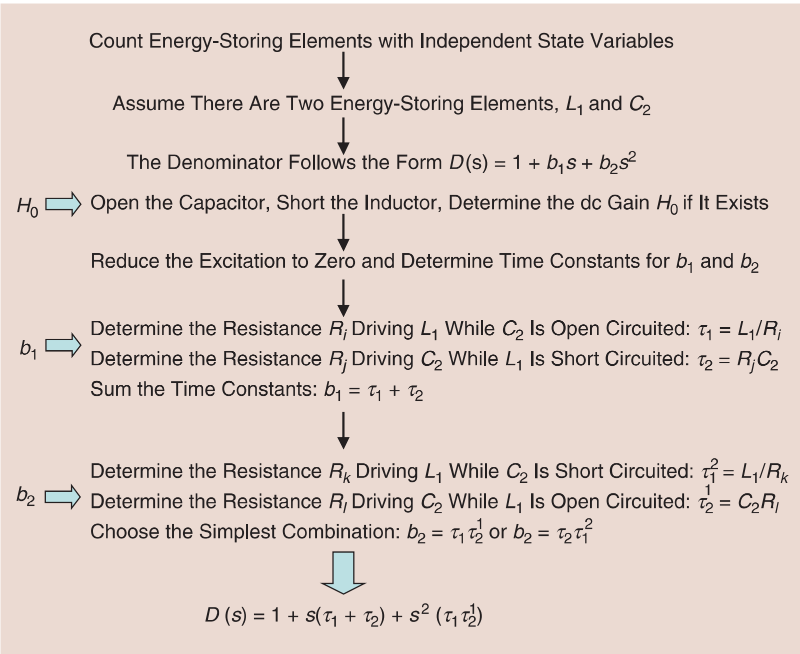

FIG 2 This flowchart explains the methodology used to determine the network time constants.

A transfer function is a mathematical relationship linking an excitation signal, the stimulus, to a response signal resulting from that excitation. If we consider a linear time-invariant system without delay and exhibiting a quasi-static gain H_0—for instance, the linearized ideal power stage of a switching converter—its transfer function, H, linking the control signal V_{err} (the stimulus) to its output V_{out} (the response) can be expressed in the following form:

$$H(s) = \frac{V_{out}(s)}{V_{err}(s)} = H_0 \frac{N(s)}{D(s)}. \quad (1)$$

The leading term, H_0, is the gain or attenuation exhibited by the system evaluated at $s = 0$. This term would carry the transfer function unit (or dimension), if any. If both the response and excitation are expressed in volts, V_{err} and V_{out} in our case, H is unitless. The numerator, $N(s)$, hosts the zeros of the transfer function. Mathematically, zeros are the roots for which the function magnitude is zero. With FACTs, we use a mathematical abstraction to let us easily unveil these zeros. Rather than solely considering the vertical axis in the s-plane as we normally do in a harmonic analysis, $(s = j\omega)$, we will cover the entire plane, allowing for negative roots. As such, if present in the circuit, a zero will manifest itself by the nulling of the output response when the input signal is tuned to the zero angular frequency, s_z. When this happens, some impedance in the transformed circuit blocks the signal propagation, and the response is nulled despite the presence of an excitation source: a series impedance in the signal path becomes infinite or a branch shunts the stimulus to the ground when the transformed circuit is excited at $s = s_z$. Note that this convenient mathematical abstraction offers tremendous help in finding the zeros by inspection, often without writing a line of algebra in passive networks. Figure 1 offers a simple flowchart that details the procedure. More details on this approach can be found in [8].

The denominator, $D(s)$, is formed by associating together the circuit natural time constants. These time constants are obtained by setting the stimulus signal to zero and determining the resistance seen from the considered capacitor or inductor terminals when temporarily removed from the circuit. By "seeing" this resistance, you imagine placing an ohmmeter across the pads of the removed energy-storing element (C or L) and read the resistance it displays. This is actually quite a simple exercise, as detailed by the flowchart in Figure 2.

Look at Figure 3, which describes a first-order passive circuit involving an injection source—the stimulus—biasing the left side of the network. The input signal V_{in} propagates through meshes and nodes to form the response V_{out} observed across the resistor R_3. We are interested in deriving the transfer function G linking V_{out} to V_{in}.

By using this technique, you will appreciate how quickly and intuitively one can determine a particular transfer function.

To determine the time constant of this example circuit, we will set the excitation to zero (a 0-V source is replaced by a short circuit, while a 0-A current source would be replaced by an open circuit) and remove the capacitor. Then, we connect mentally an ohmmeter to determine the resistance offered by the capacitor terminals. Figure 4 guides you in these steps.

If you run the exercise in Figure 4, you see r_C in series with the parallel combination of R_3 with the series-parallel arrangement of R_4 and $R_1 - R_2$. The time constant of this circuit is simply the product of R and C_1

$$\tau_1 = [r_C + (R_4 + R_1 || R_2) || R_3] C_1. \quad (2)$$

We can show that the pole of a first-order system is the inverse of its time constant. Therefore,

$$\omega_p = \frac{1}{\tau_1} = \frac{1}{[r_C + (R_4 + R_1 || R_2) || R_3] C_1}. \quad (3)$$

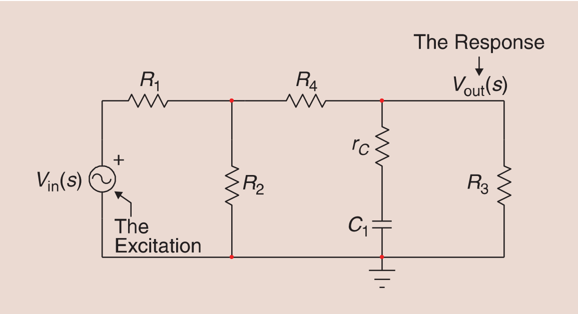

FIG 3 Determining the time constant of a circuit requires you to set the excitation to zero and look at the resistance offered by the energy-storing elements temporarily removed from the circuit.

Now, what is the quasi-static gain of this circuit for $s = 0$? In dc conditions, a capacitor becomes an open circuit, while an inductor becomes a short circuit. Apply this concept to the Figure 3 circuit, and redraw it as shown in Figure 5. In your head, you cut the connection before R_4, and you see a resistive divider involving R_1 and R_2. The Thévenin voltage across R_2 is

$$V_{th} = V_{in} \frac{R_2}{R_1 + R_2}. \quad (4)$$

The output resistance R_{th} is R_1 paralleled with R_2. The complete transfer function, therefore, involves the resistive divider made of R_4 in series with R_{th} and loaded by R_3. The value of r_C is off picture since capacitor C_1 is removed in this dc analysis. You can, therefore, write

$$G_0 = \frac{V_{out}}{V_{in}} = \frac{R_2}{R_2 + R_1} \frac{R_3}{R_4 + R_3 + R_1 \| R_2}. \quad (5)$$

We are almost there, and we are missing the zeros. As we wrote in the section "A Quick Introduction to Fast Analytical Techniques," a zero manifests itself in a circuit by blocking the propagation of the excitation signal and creating an output null (see Figure 1). If we consider a transformed circuit—in which C_1 is replaced by $1/sC_1$ (as shown in Figure 6)—what particular condition would imply a nulled response when a stimulus biases the network? Having a nulled response simply means that the current circulating in R_3 is zero. This is not a short circuit but rather a virtual ground if you prefer the analogy.

If we have no current in R_3, then the series connection of r_C and $1/sC_1$ creates a transformed short circuit:

$$Z_1(s_z) = r_C + \frac{1}{s_z C_1} = 0. \quad (6)$$

The root s_z is the zero location that we want

$$s_z = -\frac{1}{r_C C_1}, \quad (7)$$

and it leads to

$$\omega_z = \frac{1}{r_C C_1}. \quad (8)$$

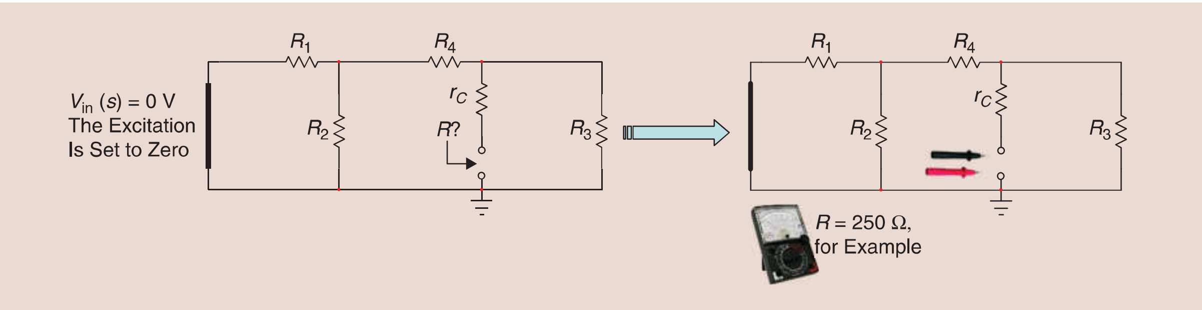

FIG 4 After replacing the 0-V source by a short circuit, you determine the resistance seen from the capacitor terminals.

We can now assemble all of these results to form the final transfer function characterizing the Figure 3 circuit

$$G(s) = \frac{R_2}{R_2+R_1}\frac{R_3}{R_4+R_3+R_1||R_2} \times \frac{1+sr_C C_1}{1+s[r_C+(R_4+R_1||R_2)||R_3]C_1} = G_0\frac{1+\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}}. \quad (9)$$

This is what is called a *low-entropy expression* in which you can immediately distinguish a quasi-static gain, G_0; a pole, ω_p; and a zero, ω_z. A high-entropy expression would be obtained by applying the brute-force approach to the original circuit when considering an impedance divider, for instance,

$$G(s) = \frac{R_2}{R_2+R_1}\frac{R_3||\left(r_C+\frac{1}{sC_1}\right)}{R_3 \parallel \left(r_C+\frac{1}{sC_1}\right)+R_4+R_1 \parallel R_2}. \quad (10)$$

Not only could you make mistakes in deriving the expression, but formatting the result in something like in (9) would require more energy. Also, note that in this particular example, we did not write a single line of algebra when writing (9). Should we later identify a mistake, it would be easy to come back to one of the individual drawings and fix it separately. The correction in (9) would then be simple. Try to run the same correction in (10), and you will probably start from scratch.

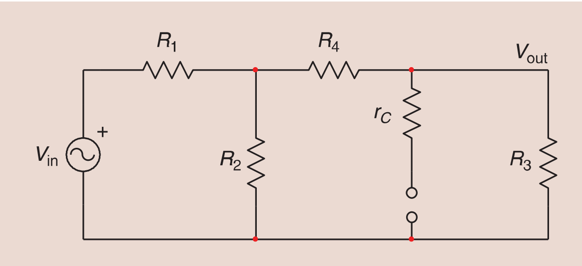

FIG 5 You open the capacitor in dc and calculate the transfer function of this simple resistive arrangement.

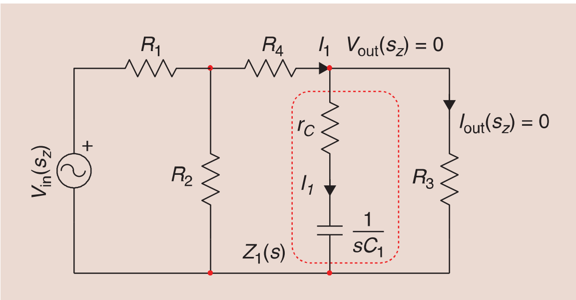

FIG 6 In this transformed circuit, when the series connection of r_C and C_1 becomes a transformed short circuit, the response disappears, and no current flows in R_3.

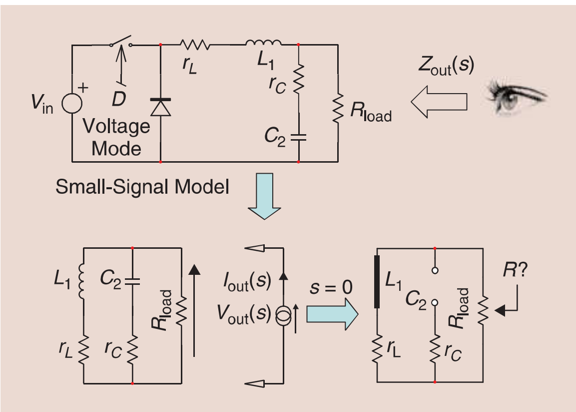

FIG 7 The determination of the CCM-operated buck converter output impedance is a good example of how FACTs simplify analyses.

FACTs Applied to a Second-Order System

FACTs work equally well for nth-order passive or active circuits. You determine the order of a circuit by counting the number of energy-storing elements whose state variables are independent. If we consider a second-order system, H, featuring a finite quasi-static gain, H_0, its transfer function can be expressed the following way:

$$H(s) = H_0\frac{1+a_1 s+a_2 s^2}{1+b_1 s+b_2 s^2}. \quad (11)$$

As H_0 carries the unit of the transfer function, the ratio made of N over D is unitless. This implies that the unit for a_1 and b_1 is time, s. You sum up the circuit time constants determined when the response is nulled for a_1 and when the excitation is zeroed for b_1. For the second-order coefficients, a_2 or b_2, the dimension is time squared, s^2, and you combine time constants in a product. However, in this time constants product, you reuse one of the time constants already determined for a_1 or b_1, while the second time constant determination requires a different notation

$$\tau_2^1 \text{ or } \tau_1^2. \quad (12)$$

In this definition, you set the energy-storing element whose label appears in the "exponent" in its high-frequency state (a capacitor is replaced by a short circuit, while an inductor would be replaced by an open circuit), and you determine the resistance seen from the second element terminals when it is temporarily removed from the circuit (subscripted reference). You carry this exercise for a nulled output when a_2 must be obtained and when the excitation is reduced to zero for b_2. Of course, when inspection works, it is always the fastest and most efficient way to obtain N. It may be a bit mysterious at first sight but nothing insurmountable, as will be demonstrated.

Figure 7 depicts a classical second-order filter involved in the determination of the output impedance of a voltage-mode buck converter operated in the continuous conduction mode (CCM). An impedance is a transfer function linking an excitation signal I_{out} to a response signal V_{out}. Here, I_{out} is the test generator that we have installed, while V_{out} is the resulting voltage produced across its terminals. To determine the various coefficients from (11), we can follow the Figure 2 flowchart and start with $s = 0$: short the inductor and open the capacitor as shown in the figure. The circuit is simple, and the resistance, R_0, seen from the current source is simply the parallel combination of r_L and R_{load}

$$R_0 = r_L \parallel R_{\text{load}}. \tag{13}$$

Do we have zeros in this circuit? We examine the transformed circuit shown in Figure 8. Let's check what component combinations would bring the response V_{out} to zero when the excitation current I_{out} is tuned at a zero angular frequency s_z. We can identify two transformed short circuits involving r_L–L_1 and r_c–C_2.

The roots for these two impedances are immediately determined.

$$r_L + sL_1 = 0 \rightarrow s_{z_1} = -\frac{r_L}{L_1}, \tag{14}$$

$$r_C + \frac{1}{sC_2} = 0 \rightarrow s_{z_2} = -\frac{1}{r_C C_2}. \tag{15}$$

The denominator $N(s)$ is, therefore, expressed by

$$N(s) = \left(1 + s\frac{L_1}{r_L}\right)(1 + sr_C C_2). \tag{16}$$

The first coefficient, b_1, of the denominator $D(s)$ is obtained by looking at the resistance offered by L_1's terminals while C_2 is in its dc state (open): you have τ_1. Then, you look at the resistance driving C_2 while L_1 is set in its dc state (short circuit): you obtain τ_2. As illustrated by Figure 9, the sketch immediately leads to the definition of b_1

$$b_1 = \tau_1 + \tau_2 = \frac{L_1}{r_L + R_{\text{load}}} + C_2[(r_L \parallel R_{\text{load}}) + r_C]. \tag{17}$$

The second-order coefficient b_2 is determined by using the notation introduced in (12). Either L_1 is set in its high-frequency state (open circuit) and you look at the resistance driving C_2 to obtain τ_2^1, or C_2 is put in its high-frequency state (short circuit) and you look at the resistance driving L_1 for τ_1^2. Figure 10 shows the two possible arrangements. You usually select the one leading to the simplest expression or the one avoiding a product indeterminacy if any ($\infty \times 0$ or ∞/∞, for instance). The following two definitions for b_2 are identical, and the first one is found to be the simplest:

$$b_2 = \tau_1\tau_2^1 = \frac{L_1}{r_L + R_{\text{load}}} C_2(r_c + R_{\text{load}})$$
$$b_2 = \tau_2\tau_1^2 = C_2[r_L \parallel R_{\text{load}} + r_C]\frac{L_1}{r_L + R_{\text{load}} \parallel r_C}. \tag{18}$$

We now have all of the ingredients to assemble the final transfer function that is defined as

$$Z_{\text{out}}(s) = (r_L \parallel R_{\text{load}})\frac{\left(1 + s\frac{L_1}{r_L}\right)(1 + sr_C C_2)}{1 + s\left(\frac{L_1}{r_L + R_{\text{load}}} + C_2[r_L \parallel R_{\text{load}} + r_C]\right) + s^2\left(L_1 C_2 \frac{r_C + R_{\text{load}}}{r_L + R_{\text{load}}}\right)}. \tag{19}$$

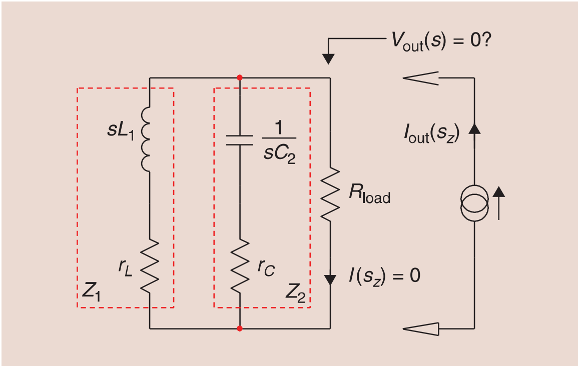

FIG 8 If impedances Z_1 or Z_2 become transformed shorts, the response V_{out} is nulled.

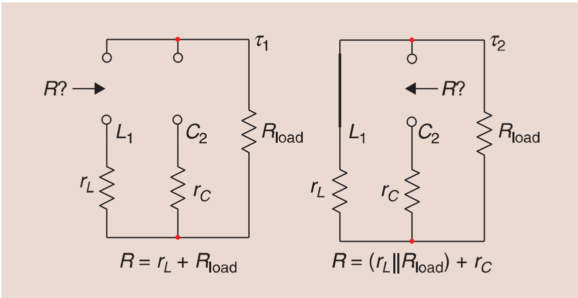

FIG 9 What resistance do you see between the selected component terminals when the second is set in its dc state?

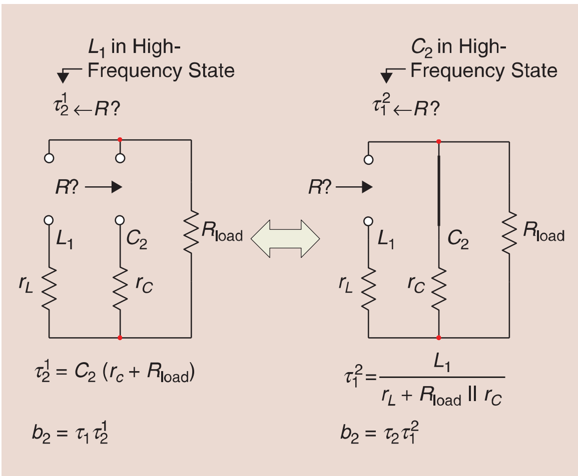

FIG 10 What resistance do you see between the selected component terminals when the second is set in its high-frequency state?

We have determined this transfer function without writing a line of algebra, just by splitting the circuit into several simple sketches, such as those of Figures 8–10, individually solved. Furthermore, as expected, (19) is already in a canonical form, and you can easily see the presence of a quasi-static gain, two zeros and a second-order denominator that you could further rearrange with a resonant term ω_0 and a quality factor Q. There is no way that we could have obtained this result this quickly considering the parallel combination of Z_1, Z_2, and R_{load}.

Deriving transfer functions by inspection is a possibility offered by FACTs in particular with passive networks. As the circuit complicates and includes voltage- or current-controlled sources, inspection becomes less obvious, and you need to resort to classical mesh and node analysis. However, FACTs offers several advantages: as you split the circuit into small individual sketches used to determine the coefficients of the final polynomial form, you can always come back to a particular drawing and individually correct it in case you have found a mistake in the final expression. Also, as you determine the terms associated with the a_i and b_i of the transfer function, you naturally end up with a polynomial form without investing further energy to collect and rearrange the terms. Finally, as shown in [4], SPICE can be of great help to verify your individual poles and zeros calculations in the presence of complicated passive and active circuits.

A DCM-Operated SEPIC with Coupled Inductors

The SEPIC is a popular structure used in applications where the output voltage must be smaller or larger than

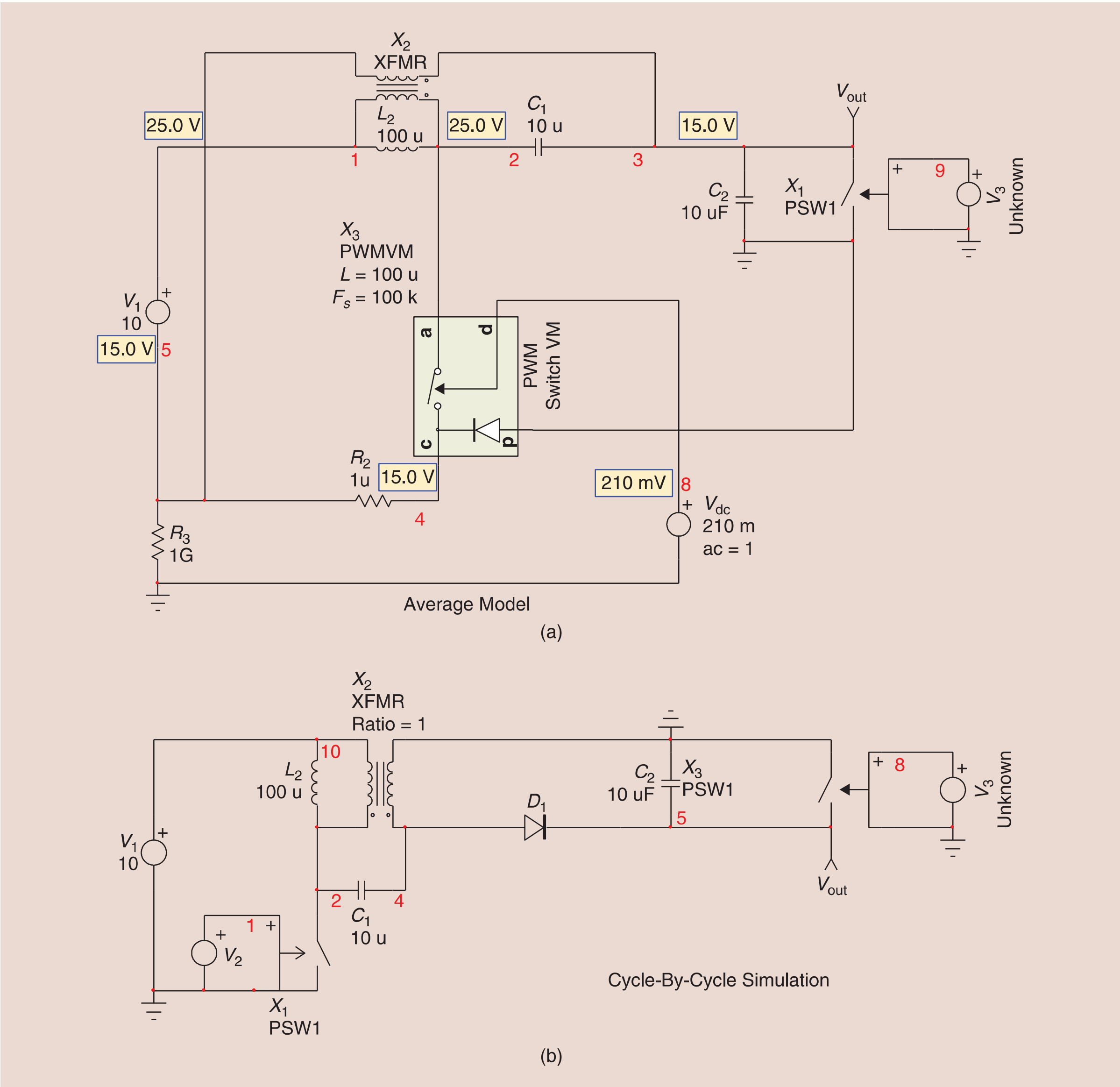

FIG 11 Two examples of the SEPIC: (a) the average model and (b) the cycle-by-cycle approach. XFMR and PSW1 are subcircuit names.

the input without sacrificing the polarity, as with a buck-boost converter. The SEPIC can be operated with coupled or uncoupled inductors in CCM or DCM. The benefits of the coupled inductors are explored in [9] and will not be discussed here. Our interest lies in determining the control-to-output transfer function of the coupled-inductor SEPIC when operated in DCM. Figure 11 represents the autotoggling, voltage-mode control PWM switch described in [10] and connected in a SEPIC configuration. The load is purposely reduced to force DCM. A transient step is applied after the start-up sequence is completed. A cycle-by-cycle circuit is captured and simulated in similar operating conditions.

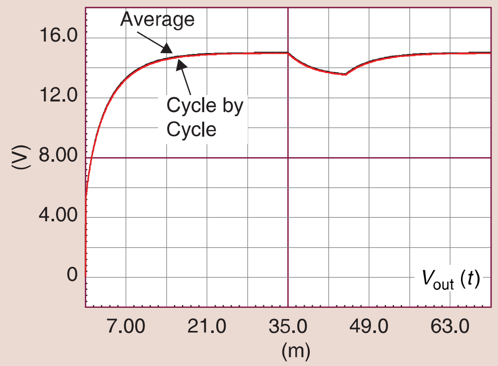

FIG 12 The average model transient response exactly matches that of the cycle-by-cycle model.

A simulation is run to compare the output responses of both circuits. As confirmed by Figure 12, the two responses are extremely close to each other. The left side of the curve describes the startup sequence, while the right-side section shows how both models react to the load step. Having identical responses at this stage is a first indication that the large-signal model, on average, properly mimics the SEPIC internals and we can proceed with the small-signal version.

The large-signal model of the DCM PWM switch is replaced by its small-signal version derived in [10], which differs from that described in [5]. Both models lead to identical analyses; however, V. Vorpérian in [5] considered a common-common configuration (terminal *c* is grounded), while I kept the original common-passive configuration for the sake of building an autotoggling DCM-CCM model. The schematic updated with the small-signal model of the DCM PWM switch appears in Figure 13. The right-side parameters list computes all the *k*-coefficients needed for the analysis.

Determining the Quasistatic Gain

To determine the quasi-static gain, you short all the inductors and open all the capacitors as detailed in Figure 2.

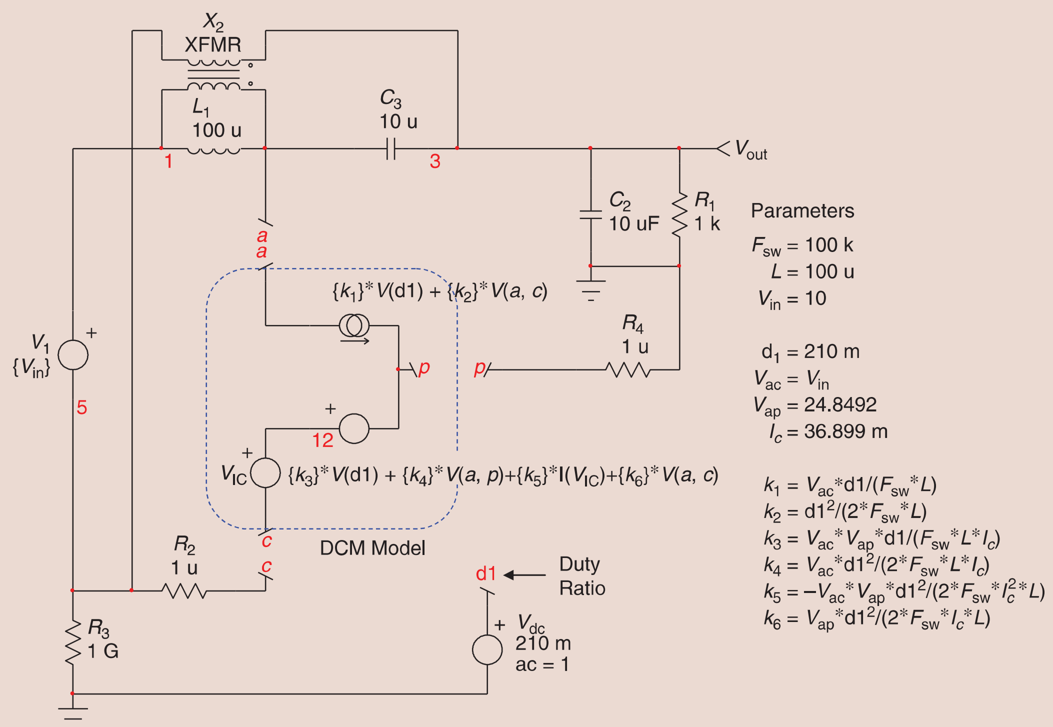

FIG 13 This is the small-signal model of the SEPIC operated in DCM. Node d1 is the duty ratio bias and the injection point. All small-signal coefficients are automated in the parameters window.

This is exactly what SPICE does when calculating an operating bias point. Then, you rearrange all the sources and components to simplify the circuit and make it look friendlier for the analysis. When you do this, I recommend always running a sanity check, confirming that the dynamic response delivered by the new circuit perfectly matches that from Figure 13. Any deviation indicates that you made a mistake or the assumption in the simplification was too optimistic: redo the exercise until a perfect match in magnitude and phase is obtained. The circuit of Figure 14 is assembled.

A few lines of algebra will lead us to the output voltage expression

$$\frac{V_{out}}{R_{load}} = I_C - \frac{(V_{(a)} - V_{(c)})D^2}{2F_{sw}L_1} \rightarrow I_C = \frac{V_{out}}{R_{load}} + \frac{V_{in}D^2}{2F_{sw}L_1} \tag{20}$$

$$V_{out} = \frac{(V_{out}+V_{in})(V_{out}+V_{in}-V_{out})D^2}{2F_{sw}I_C L_1} = \frac{(V_{out}+V_{in})V_{in}D^2}{2F_{sw}I_C L_1}. \tag{21}$$

Substitute I_c from (20) in (21), and solve for V_{out}. You should obtain

$$V_{out} = DV_{in}\sqrt{\frac{1}{2\tau_L}} \text{ with } \tau_L = \frac{L_1}{R_{load}T_{sw}}. \tag{22}$$

The small-signal quasi-static gain is simply

$$H_0 = \frac{dV_{out}(D)}{dD} = \frac{d}{dD}\left(DV_{in}\sqrt{\frac{1}{2\tau_L}}\right) = V_{in}\sqrt{\frac{1}{2\tau_L}}. \tag{23}$$

Time Constants Determination

Rather than solving the whole transfer function at once with the complete schematic of Figure 13, we will use the FACTs and individually determine the time constants of the circuit. This method offers the advantage of confronting the result that you obtain with a SPICE simulation of the individual sketch. This is of tremendous help to progress step-by-step and track errors before realizing the final result is wrong after hours of work!

To determine the time constant, the excitation is reduced to zero (check Figure 2). Here, because we want the control-to-output transfer function, the excitation is d_1. Reducing it to zero helps simplify the circuit as shown in Figure 15.

There are a few equations that we can write to describe this circuit recognizing that $I_C = I_T$:

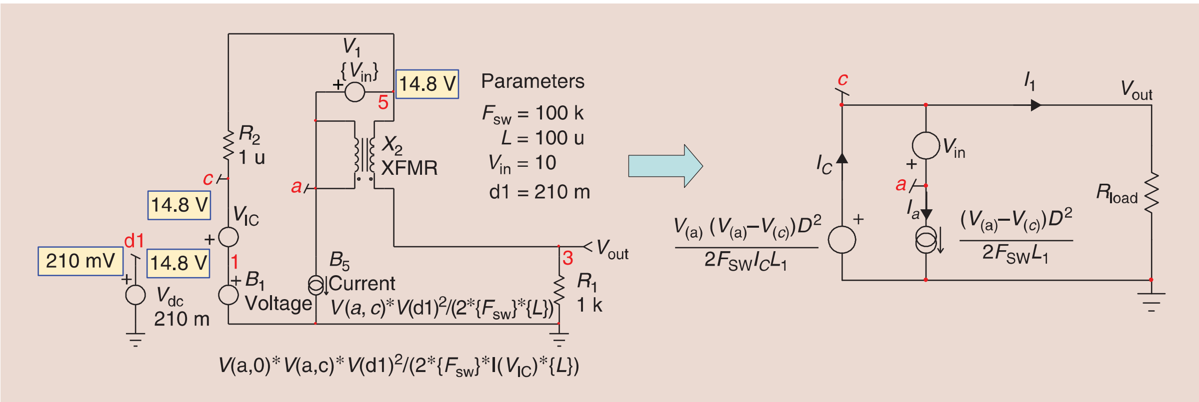

FIG 14 This is the final dc circuit used to determine the quasi-static gain H_0.

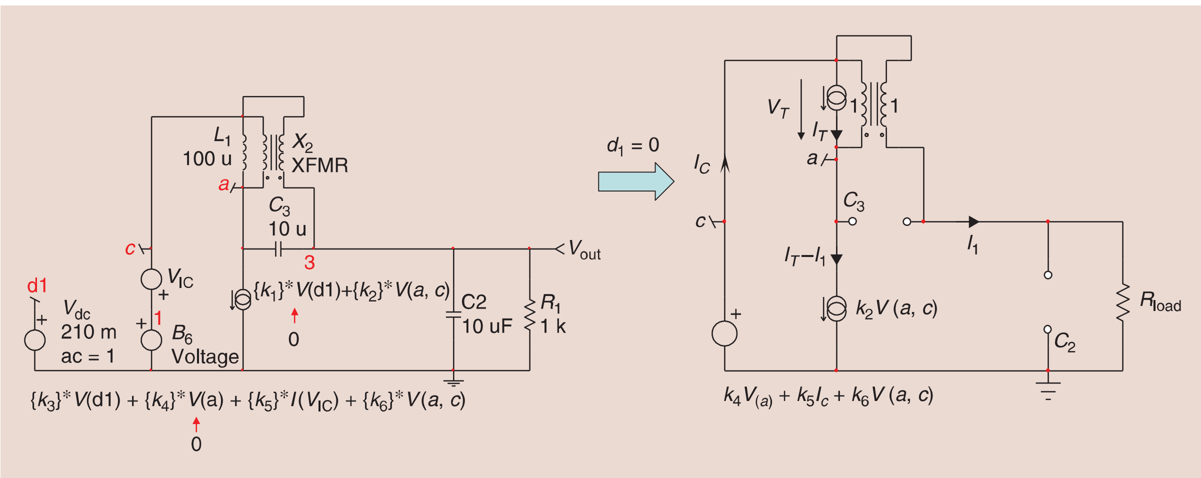

FIG 15 The reduction of the excitation to zero helps to simplify the circuit. Here, we start with the resistance driving the inductor L_1.

$$V_T = V_{(a)} - V_{(c)}, \tag{24}$$

$$V_{(a)} = R_{\text{load}} I_1, \tag{25}$$

$$I_T - I_1 = k_2(V_{(a)} - V_{(c)}) \rightarrow V_{(a)} = \frac{I_T + V_{(c)} k_2}{k_2 + \dfrac{1}{R_{\text{load}}}}, \tag{26}$$

$$V_{(c)} = k_4 V_{(a)} + k_5 I_C + k_6 V_{(a)} - k_6 V_{(c)}. \tag{27}$$

You substitute (26) in (27) and then solve for $V_{(c)}$. Substitute $V_{(c)}$ in (26), and solve for $V_{(a)}$. Then, write

$$\frac{V_T}{I_T} = \frac{V_{(a)} - V_{(c)}}{I_T} = \frac{R_{\text{load}}(1-k_4) - k_5}{k_6 + R_{\text{load}} k_2 (1-k_4) + 1}. \tag{28}$$

If you rearrange and replace the k coefficients by their definition from Figure 13, you have the definition for time constant τ_1

$$\tau_1 = \frac{L_1}{\dfrac{R_{\text{load}}(1-k_4) - k_5}{k_6 + R_{\text{load}} k_2 (1-k_4) + 1}} = \frac{L_1}{\dfrac{R_{\text{load}}}{M(1+M)+0.5}}. \tag{29}$$

The second time constant implies looking at the resistance seen from C_2's terminals while L_1 is a short circuit. The new circuit appears in Figure 16. As L_1 shorts terminals a and c together, simplification occurs updating the circuit to that of the right side of the picture.

Again, a few simple equations will lead you to the result quickly:

$$V_T = (I_T + I_C) R_{\text{load}} \rightarrow I_C = \frac{V_T - I_T R_{\text{load}}}{R_{\text{load}}}, \tag{30}$$

$$V_T = k_4 V_T + k_5 I_C \rightarrow V_T = \frac{k_5 I_C}{1-k_4}. \tag{31}$$

Substitute (30) in (31), then solve for V_T, and rearrange. You should find

$$\frac{V_T}{I_T} = \frac{k_5}{k_4 + \dfrac{k_5}{R_{\text{load}}} - 1} = \frac{R_{\text{load}}}{2} \rightarrow \tau_2 = \frac{R_{\text{load}}}{2} C_2. \tag{32}$$

If you try to determine the third time constant involving C_3, the transformer configuration (perfect coupling) makes its terminals voltage equal to 0 V: the capacitor plays no role in the dynamic transfer function. The first coefficient b_1 is thus defined by

$$b_1 = \tau_1 + \tau_2 = \frac{L_1}{\dfrac{R_{\text{load}}}{M(1+M)+0.5}} + \frac{R_{\text{load}}}{2} C_2 \approx \frac{R_{\text{load}}}{2} C_2. \tag{33}$$

The Second-Order Coefficient

For the second-order coefficient, we will set capacitor C_2 in its high-frequency state (replace it by a short circuit) while we determine the resistance driving inductor L_1. The drawing illustrating this approach is given in Figure 17. Because the output is shorted by C_2, nodes a and c are at the same 0-V potential. The electric circuit is simplified to that of the right-side sketch.

We can write a first equation describing the V_T voltage. Observing that 1) I_T and I_C are identical and 2) $V_T = -V_{(c)}$, we have

$$V_T = -(k_5 I_C - k_6 V_{(c)}) = -(k_5 I_T + k_6 V_T) \rightarrow V_T(1+k_6) = -k_5 I_T. \tag{34}$$

Factoring V_T/I_T, the resistance seen from L_1's terminals is

$$\frac{V_T}{I_T} = \frac{k_5}{-1-k_6}. \tag{35}$$

The second-order time constant τ_1^2 is defined by

$$\tau_1^2 = \frac{L_1}{\left(-\dfrac{k_5}{1+k_6}\right)} = \frac{L_1}{\dfrac{R_{\text{load}} V_{\text{in}}^2}{(V_{\text{in}} + V_{\text{out}})^2}} = \frac{L_1}{R_{\text{load}}\left(\dfrac{1}{1+M}\right)^2}. \tag{36}$$

If we consider that $V_{\text{out}} = M V_{\text{in}}$, the b_2 coefficient is expressed as

$$b_2 = \tau_2 \tau_1^2 = \frac{L_1 C_2 (1+M)^2}{2}. \tag{37}$$

Assembling the time constants, we have determined leads to the denominator $D(s)$

$$D(s) = 1 + b_1 s + b_2 s^2 = 1 + s(\tau_1 + \tau_2) + s^2 \tau_2 \tau_1^2. \tag{38}$$

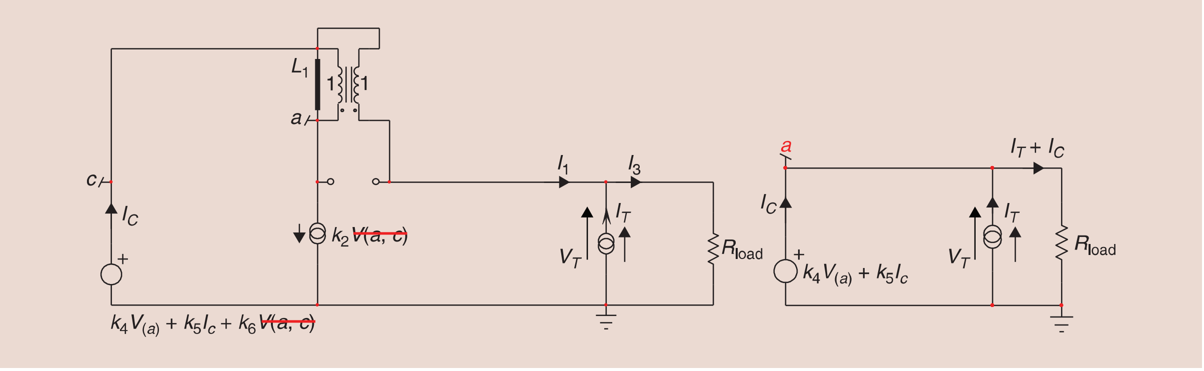

FIG 16 Shorting the inductor truly simplifies the circuit.

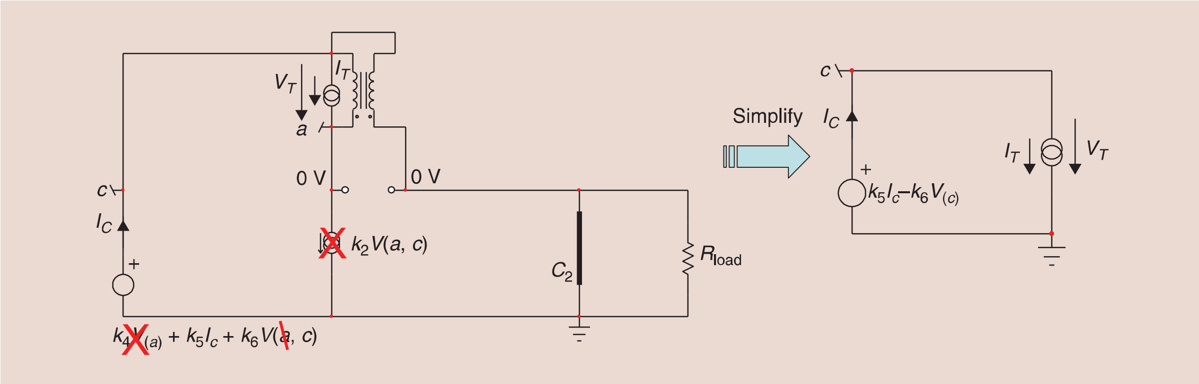

FIG 17 The second-order coefficient sets one of the energy-storing elements in its high-frequency state, C_2, while you determine the resistance seen from inductor terminals.

If we consider a low-Q approximation, this second-order denominator can be approximated by two cascaded poles defined as

$$\omega_{p_1} = \frac{1}{b_1} = \frac{1}{\tau_1 + \tau_2} \approx \frac{1}{\tau_2} \rightarrow \omega_{p_1} = \frac{2}{R_{load}C_2} \tag{39}$$

$$\omega_{p_2} = \frac{b_1}{b_2} = \frac{\tau_1 + \tau_2}{\tau_2\tau_1^2} \approx \frac{1}{\tau_1^2} \rightarrow \omega_{p_2} = \frac{R_{load}}{L_1}\left(\frac{1}{1+M}\right)^2 \tag{40}$$

and combined as

$$D(s) \approx \left(1 + \frac{s}{\omega_{p_1}}\right)\left(1 + \frac{s}{\omega_{p_2}}\right). \tag{41}$$

Determination of the Zero

As explained at the beginning of this article, when the excitation is tuned to the zero angular frequency s_z, the response of the transformed network is nulled (see Figure 1). The exercise will now consist of bringing the excitation back and determining the condition in the transformed circuit that creates an output null. Figure 18 shows the updated circuit that we need to study. The interesting thing with the output null is its propagation to other nodes. For instance, if $V_{out} = 0$ V, then, because of the transformer upper connection, node a is also at 0 V, and all expressions involving this node can be simplified

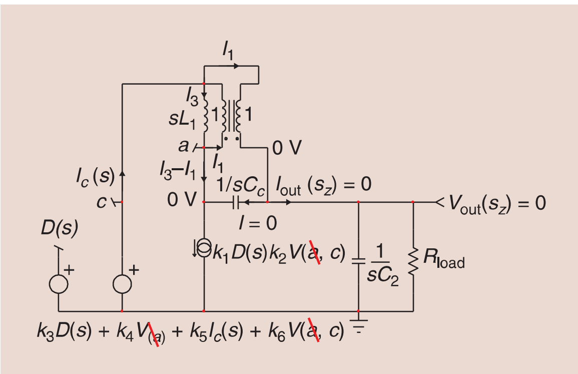

FIG 18 A specific condition in the transformed network observed at $s = s_z$ nulls the response.

as shown in the figure. If the output is nulled, then current I_1 is also null, implying that $I_c = I_3$.

The voltage of node c is defined by

$$V_{(c)}(s) = \frac{D(s)k_3 + I_c(s)k_5}{1 + k_6}. \tag{42}$$

The current I_c is therefore equal to the voltage at node c divided by the impedance of L_1

$$I_c(s) = \frac{\dfrac{D(s)k_3 + I_c(s)k_5}{1 + k_6}}{sL_1} \rightarrow I_c(s) = \frac{D(s)k_3}{sL_1(1 + k_6) - k_5}, \tag{43}$$

while current I_3 is equal to

$$I_3(s) = k_1D(s) - k_2V_{(c)} = k_1D(s) - k_2sL_1I_c(s). \tag{44}$$

Now, substitute (43) in (44), and then equate I_c and I_3:

$$k_1D(s) - k_2sL_1\frac{D(s)k_3}{sL_1(1 + k_6) - k_5} = \frac{D(s)k_3}{sL_1(1 + k_6) - k_5}. \tag{45}$$

Solve for s, replace the k-coefficients by their values from Figure 13, rearrange, and you find

$$s_z = \frac{R_{load}}{L_1M(1 + M)}. \tag{46}$$

This is a positive root and, therefore, a right half-plane zero. The complete transfer function is obtained by gathering all the pieces and recognizing that poles and zeros are actually those of a DCM buck-boost converter:

$$H(s) = H_0\frac{1 - \dfrac{s}{\omega_z}}{\left(1 + \dfrac{s}{\omega_{p_1}}\right)\left(1 + \dfrac{s}{\omega_{p_2}}\right)}, \tag{47}$$

with

$$\omega_{p_1} = \frac{2}{R_{load}C_2} \tag{48}$$

$$\omega_{p_2} = \frac{R_{load}}{L_1}\left(\frac{1}{1+M}\right)^2 \tag{49}$$

$$\omega_z = \frac{R_{load}}{L_1M(1 + M)}, \tag{50}$$

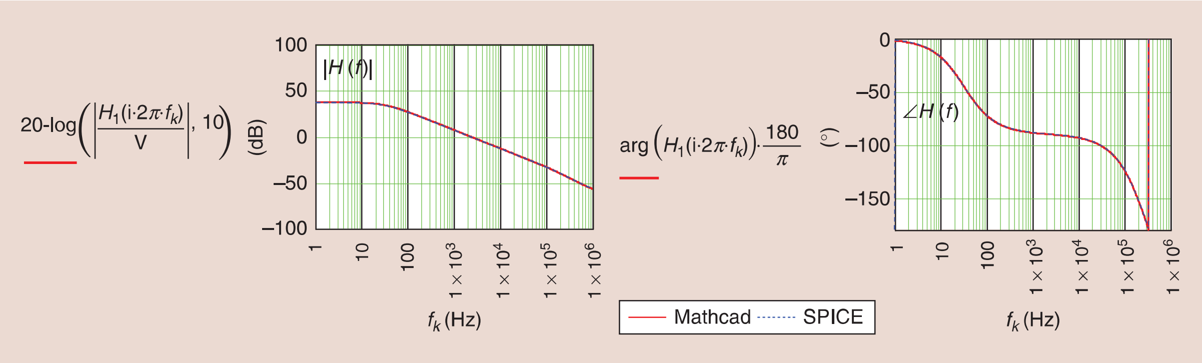

FIG 19 Both Mathcad and SPICE deliver the same response (curves perfectly superimpose).

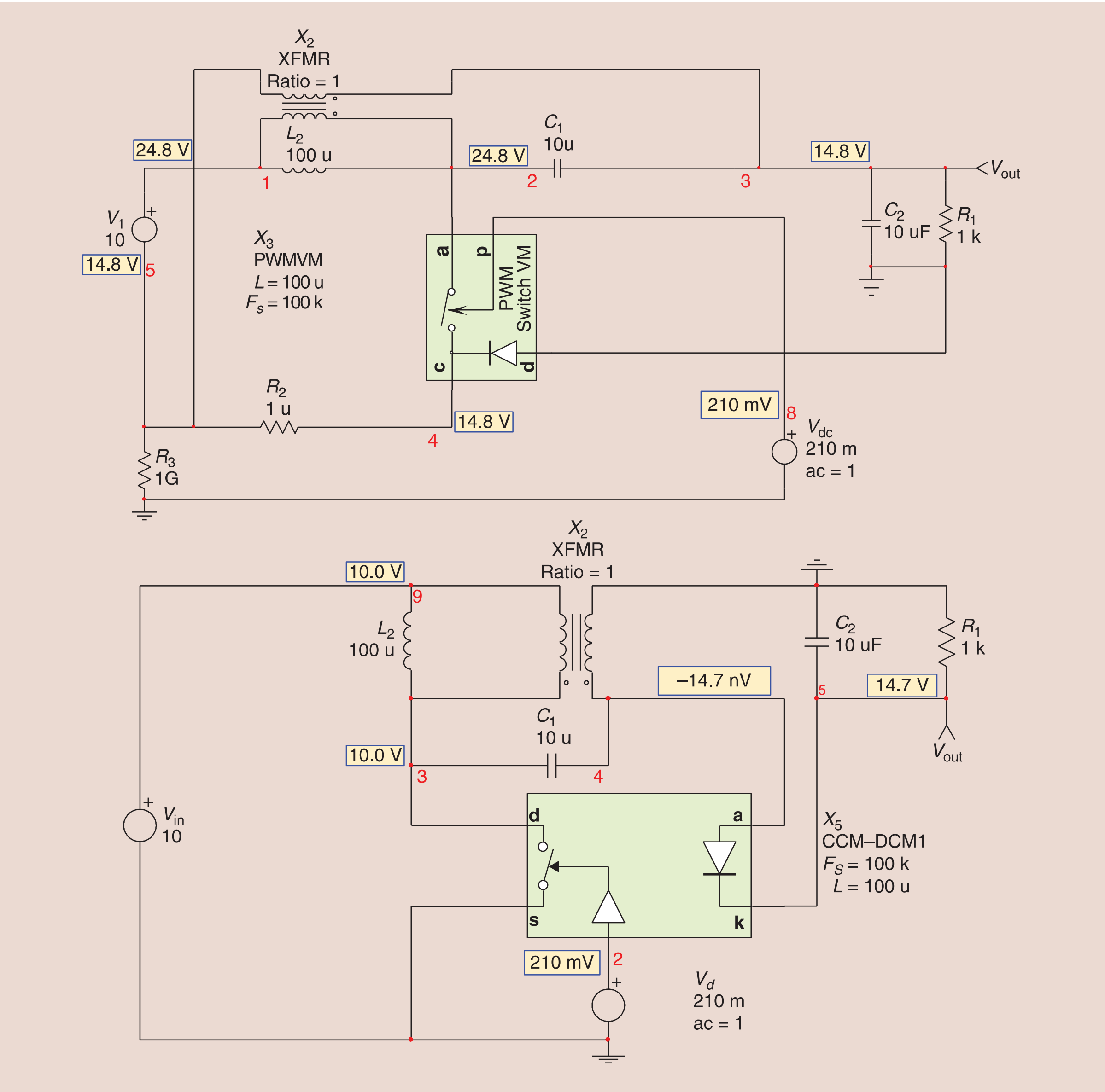

FIG 20 Colorado Power Electronics Center (CoPEC) average models include separate connections for the switch and the diode.

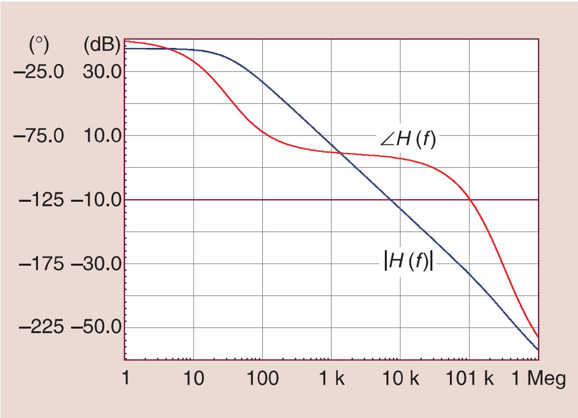

FIG 21 The DCM PWM switch and the CoPEC DCM model give identical dynamic responses.

and

$$H_0 = V_{\text{in}}\sqrt{\frac{1}{2\tau_L}}. \quad (51)$$

As a final check, we can compare the dynamic response delivered by Mathcad and that of the SPICE simulation from the Figure 11 large-signal model. As shown in Figure 19, the curves are perfectly matching.

Another verification consists of simulating the same SEPIC structure with a different average model whose construction is detailed in [11]. This is also an autotoggling CCM-DCM model, but it is wired in a slightly different way. Figure 20 shows both the average models in a similar SEPIC configuration.

Figure 21 confirms that both ac responses in phase and magnitude are perfectly identical.

Conclusions

FACTs offer a fast and efficient method to derive the transfer function of linear circuits. With passive networks, inspection is possible, and very often, a transfer function can be obtained without writing a single line of algebra. As circuits complicate and include active sources, you have to resort to classical Kirchhoff's current law and Kirchhoff's voltage law analyses. However, as you determine individual polynomial factors in the numerator and the denominator, it is easy to track errors and focus only on the defective term, if any. The help of small sketches and SPICE to that matter is invaluable with complicated networks. Finally, the end result comes out in a meaningful format and offers immediate insight on where poles and zeros are located. This is of utmost importance, as you must know where the offenders hide in the transfer function. As a designer, you must neutralize them so that natural production spreads, or component changes do not jeopardize the stability of your system during its operating life.

About the Author

Christophe Basso (christophe.basso@onsemi.com) received his B.S.E.E. degree equivalent in electrical engineering from Montpellier University, France, and his M.S.E.E. degree in electrical engineering from Institut National Polytechnique of Toulouse, France. He is a technical fellow of ON Semiconductor, based in Toulouse, where he leads an application research and development team dedicated to developing new offline pulsewidth modulation controller specifications. He has more than 20 years of power supply industry experience and has originated numerous integrated circuits for ON Semiconductor, including the NCP120X series, which set new standards for low standby power converters. He has published several books on switching power supplies; his latest is *Linear Circuit Transfer Functions: An Introduction to Fast Analytical Techniques*. Prior to joining ON Semiconductor, he was an application engineer at Motorola Semiconductor in Toulouse, France, and a power supply designer at the European Synchrotron Radiation Facility in Grenoble, France. He holds 17 patents on power conversion. He is a Senior Member of the IEEE.

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